

# Long-term classical and general relativistic effects on the radial velocities of the stars orbiting Sgr A\*

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## ABSTRACT

We analytically work out the cumulative, i.e. averaged over one orbital revolution, time variations  $\langle \dot{v}_\rho \rangle$  of the radial velocity  $v_\rho$  of a typical S star orbiting the Supermassive ( $M_\bullet \approx 10^6 M_\odot$ ) Black Hole (SBH) hosted by the Galactic Center (GC) in Sgr A\* caused by several dynamical effects. They are the general relativistic gravitoelectromagnetic (GEM) fields of the SBH, its quadrupole mass moment  $Q_2$ , and a diffuse dark matter distribution around the SBH. All of them induce non-zero long-term radial accelerations proportional to the eccentricity  $e$  of the orbit. By taking the S2 star, orbiting the SBH along a highly eccentric ( $e = 0.8831$ ) ellipse with a period  $P_b = 15.9$  yr and semi-major axis  $a = 1031.69$  au, we numerically compute the magnitudes of its radial accelerations. The largest effects are due to the general relativistic Schwarzschild-like gravitoelectric (GE) field, with  $\langle \dot{v}_\rho^{(\text{GE})} \rangle = 8.2 \times 10^{-5} \text{ m s}^{-2}$ , and the diffuse material distribution, modeled with a Plummer-type mass density profile, with  $\langle \dot{v}_\rho^{(\text{dm})} \rangle = 3.8 \times 10^{-6} \text{ m s}^{-2}$ . The effects caused by the general relativistic Kerr-type gravitomagnetic (GM) field and by  $Q_2$  are smaller by orders of magnitude. By assuming an uncertainty in measuring the radial velocities of about  $15 \text{ km s}^{-1}$ , the future accuracy in measuring  $\langle \dot{v}_\rho \rangle$  can be evaluated to be of the order of  $2.4 \times 10^{-5} \text{ m s}^{-2}$  over an observational time span  $\Delta t = 20$  yr. Currently, the available radial velocity measurements cover just 7 yr.

*Subject headings:* black hole physics-Galaxy:center-relativity-techniques: radial velocities

## 1. Introduction

It is now widely accepted that the Galactic Center (GC) hosts a Supermassive Black Hole (SBH) (Wollman et al. 1977; Genzel et al. 1996; Ghez et al. 2008; Schödel et al. 2002; Melia 2007), whose position coincides with that of the radio-source Sagittarius A\* (Sgr A\*) (Reid et al. 2007). The SBH has a mass of the order of  $M_{\bullet} = 4 \times 10^6 M_{\odot}$  (Ghez et al. 2008; Gillessen et al. 2009a,b) and, consequently, a Schwarzschild radius  $r_g = 0.084$  au. In its immediate vicinity a number of rapidly orbiting stars (Paumard et al. 2006) have been detected and tracked in the infrared since 1992 at the 8.2-m Very Large Telescope (VLT) on Cerro Paranal, Chile and the 3.58-m New Technology Telescope (NTT) on La Silla, Chile (Eckart & Genzel 1996), and since 1995 at the Keck 10-m telescope on Mauna Kea, Hawaii (Ghez et al. 1998). They are mostly main-sequence stars of spectral class B, and are naturally used as test particles for the gravitational potential in which they move. Particularly important is the bright star S2 (S0-2 in the Keck nomenclature), of spectral type B0-2 V, orbiting the SBH in 15.9 yr along an orbit with ellipticity  $e = 0.8831$  and semimajor axis  $a = 1031.69$  au (Gillessen et al. 2009a). Indeed, a complete orbital revolution of it is now covered by the available data records (Gillessen et al. 2009a,b).

If we look at the smallness of the ratio  $\tau$  of the average distance  $\langle r \rangle = a(1 + e^2/2)$  to the Schwarzschild radius  $r_g$  as an index of the importance of the Einstein’s General Theory of Relativity (GTR) in several astronomical and astrophysical systems, it can be easily realized that the Galactic SBH and its stars is, in principle, an ideal local laboratory to put on the test GTR and other alternative theories of gravity. Indeed, by considering the Earth’s artificial satellite LAGEOS (Cohen & Smith 1985), the Sun’s planet Mercury, the extrasolar planet WASP-19b (Hebb et al. 2010), the double binary pulsar PSR J0737-3039A/B (Burgay et al. 2003) and the star S2 in Sgr A\*, we have

$$\begin{aligned} \tau_{\text{LAGEOS}} &= 1.383319 \times 10^9, \\ \tau_{\text{Mercury}} &= 2.0023 \times 10^7, \\ \tau_{\text{WASP-19b}} &= 8.74 \times 10^5, \\ \tau_{\text{PSR J0737-3039A/B}} &= 1.15 \times 10^5, \\ \tau_{\text{S2}} &= 1.7 \times 10^4. \end{aligned} \tag{1}$$

It can be noted that  $\tau$  for S2 is one order of magnitude smaller than that for PSR J0737-3039A/B. In this respect, several authors (Jaroszynski 1998; Fragile & Mathews 2000; Rubilar & Eckart 2001; Weinberg et al. 2005; Kraniotis 2007; Nucita et al. 2007; Will 2008; Preto & Saha 2009; Kannan & Saha 2009; Merrit et al. 2010) worked out with a variety of techniques and approximations the direct effects of GTR on different quantities characterizing the orbital motions about the SBH, mainly some Keplerian orbital elements.

If, on the one hand, they are useful to give “intuitive” valuable insights about the magnitude of the relativistic effects occurring in such a scenario, on the other hand they are not directly measurable.

Concerning the motion of the S stars around the SBH in the GC, the directly observable quantities are the astrometric measurements of their positions in the sky in terms of right ascension  $\alpha$  and declination  $\delta$ , and their radial velocities  $v_r$ . Concerning the first kind of observations, according to Eisenhauer et al. (2009), future astrometric measurements of S2 may bring its relativistic perinigricon<sup>1</sup> precession  $\langle \dot{\omega}_\bullet \rangle$  into the measurability domain. Indeed, the perinigricon rate would be indirectly inferred from the corresponding apparent position shift. To this aim, it must also be considered that such a shift is not as easily detected as it may seem since it needs to be measured from the same data from which also the orbital elements have to be determined (Fritz et al. 2010). Thus, here we will focus on the dynamical effects directly caused by GTR and other competing classical forces on the radial velocity  $v_r$ . On the one hand, it will be possible to straightforwardly work out in an analytical way the net time variations of it averaged over one orbital revolution. This allows for a more direct and unambiguous confrontation of the theoretical predictions with the observations. On the other hand, from a practical point of view the radial velocity data are easier to handle with respect to the astrometric observations. Indeed, the inclusion of new data into pre-existent records needs no special care because the radial velocities refer to the Local Standard of Rest (LSR) (Reid et al. 2009). Instead, for the astrometric data it turns out that only an approximate realization of a common relative reference frame is possible. It implies that the exact definition of the coordinates is a matter of each data analysis in such a way that simply merging two different sets of astrometric positions would yield incorrect results (Gillessen et al. 2009b).

In our calculation, we will proceed as follows. For the sake of generality, let us assume that an explicit, analytical expression is available for a given observable  $Y$  in such a way that it is function of all or some Keplerian orbital elements, i.e.  $Y = Y(f, \{\kappa\})$ , where  $f$  is

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<sup>1</sup>Schödel et al. (2002) introduced in the scientific literature for the first time such a term for designing the pericenter  $\omega$  in the case of a BH. It comes from the Latin word “*niger*”, meaning “black”. Before, the term “perimelasma” was used by the physicist G.A. Landis (2001) in a science-fiction tale of him. Actually, the Greek word “*μέλας*” means “black”, but, on the other hand, “melasma” (*Chloasma faciei*) is a medical term denoting a skin affection which consists of a tan or dark skin discoloration. An earlier and somewhat more adequate term is “peribothron”, coming from the Greek word “*βόθρος*” for “pit”. It is due to a suggestion by W.R. Stoeger to Frank & Rees (1976). After all, BHs may not be entirely “black” since, under certain circumstances, they may radiate, so that a term recalling their deep gravitational well appears more suitable to characterize them. Let us note that the definition of BHs as “frozen stars” by the former Soviet scholars may have lead to the word “peripekton”, from the Greek word “*πηκτός*” for “frozen”.

the true anomaly, and  $\kappa$  denotes the ensemble of the Keplerian orbital elements explicitly entering  $Y$  apart from the mean anomaly  $\mathcal{M}$ . Then, we straightforwardly compute its secular variation as the sum of two parts. The first one is purely Keplerian, and it vanishes over one orbital period  $P_b$ . The second one is due to the non-Keplerian variations of all the orbital elements induced by the dynamical perturbation considered. The total result is, thus,

$$\left\langle \frac{dY}{dt} \right\rangle = \frac{\langle \Delta Y \rangle}{P_b} = \left( \frac{1}{P_b} \right) \int_0^{2\pi} \left[ \frac{\partial Y}{\partial f} \frac{df}{d\mathcal{M}} \frac{d\mathcal{M}}{dt} + \sum_{\kappa} \frac{\partial Y}{\partial \kappa} \frac{d\kappa}{dt} \right] \left( \frac{dt}{df} \right) df. \quad (2)$$

In it,  $d\mathcal{M}/dt$  and  $d\kappa/dt$  are the instantaneous variations<sup>2</sup> of the Keplerian orbital elements computed with, e.g., the Gauss variation equations and evaluated onto the unperturbed Keplerian ellipse, while  $df/d\mathcal{M}$  and  $dt/df$  are the usual Keplerian expressions for such derivatives: see eq. (10) and eq. (11) below.

The paper is organized as follows. In Section 2 we deal with some kinds of both classical and relativistic perturbing accelerations. The long-term effects caused by them on the radial velocity are analytically worked out in Section 3. In Section 4 we perform numerical calculations by using the S2 star, and confront them with the present-day measurement accuracy. Section 5 is devoted to the summarizing our findings.

## 2. The perturbing accelerations

### 2.1. General treatment and overview

Here we deal with a generic perturbing acceleration  $\mathbf{A}$  induced by a given dynamical effect which can be considered as small with respect to the main Newtonian monopole  $A_{\text{Newton}} = GM_{\bullet}/r^2$ , where  $G$  is the Newtonian constant of gravitation and  $r$  is the relative star-SBH distance. The stars orbiting the SBH are assumed test particles: their masses are about  $m_{\star} \lesssim 10^{-5} M_{\bullet}$ , and relativistic corrections to their internal structures are assumed to be too small to yield noticeable effects on their orbital motions.

First,  $\mathbf{A}$  has to be projected onto the radial, transverse and normal orthogonal unit vectors  $\hat{\mathbf{R}}, \hat{\mathbf{T}}, \hat{\mathbf{N}}$  of the co-moving frame of the test particle orbiting the central body acting as source of the gravitational field. Their components, in cartesian coordinates of a reference frame centered in the primary, are (Montenbruck & Gill 2000)

$$\hat{\mathbf{R}} = \begin{pmatrix} \cos \Omega \cos u & -\cos \lambda \sin \Omega \sin u \\ \sin \Omega \cos u & +\cos \lambda \cos \Omega \sin u \\ \sin \lambda \sin u \end{pmatrix} \quad (3)$$

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<sup>2</sup>Actually,  $d\mathcal{M}/dt$  is the sum of the Keplerian mean motion  $n$  and a non-Keplerian term, as we will see later. Its Keplerian part yields from eq. (2) the Keplerian variation of  $Y$ .

$$\hat{\mathbf{T}} = \begin{pmatrix} -\sin u \cos \Omega - \cos \lambda \sin \Omega \cos u \\ -\sin \Omega \sin u + \cos \lambda \cos \Omega \cos u \\ \sin \lambda \cos u \end{pmatrix} \quad (4)$$

$$\hat{\mathbf{N}} = \begin{pmatrix} \sin \lambda \sin \Omega \\ -\sin \lambda \cos \Omega \\ \cos \lambda \end{pmatrix}. \quad (5)$$

In eq. (3)-eq. (5),  $\Omega, \omega, \lambda$  are the longitude of the ascending node<sup>3</sup>, the argument of pericenter, reckoned from the line of the nodes<sup>4</sup>, and the inclination of the orbital plane to the reference  $\{xy\}$  plane, respectively. In general, we will choose the unit vector  $\hat{\boldsymbol{\rho}}$  of the line-of-sight, pointing from the object to the observer, to be directed along the positive  $z$  axis, so that the  $\{xy\}$  plane coincides with the usual plane of the sky. With such a choice, corresponding to the frame actually used in data reduction (Ghez et al. 2008),  $\lambda$  is the familiar  $i$ , and  $\Omega$  is an angle in the plane of the sky counted from the mean vernal point at J2000 epoch along which the reference  $x$  axis is customarily chosen; it is such a node which is actually determined from the observations (Ghez et al. 2008; Gillessen et al. 2009a,b). In other cases, in order to compute more easily certain dynamical perturbations, we will orient our frame with the  $z$  axis aligned with the central body's proper angular momentum  $\mathbf{L}$ , so that the reference  $\{xy\}$  plane will coincide with the body's equatorial plane. In this case,  $\Omega$  will be an angle lying in such a plane, and it is not the one released in literature (Ghez et al. 2008; Gillessen et al. 2009a,b). Moreover,  $u \doteq f + \omega$  is the argument of latitude. Subsequently, the projected components of  $\mathbf{A}$  have to be evaluated onto the Keplerian ellipse

$$r = \frac{p}{1 + e \cos f}, \quad p \doteq a(1 - e^2), \quad (6)$$

where  $p$  is the semilatus rectum and  $a, e$  are the semi-major axis and the eccentricity, respectively. The cartesian coordinates of the Keplerian motion in space are (Montenbruck & Gill 2000)

$$\begin{aligned} x &= r (\cos \Omega \cos u - \cos \lambda \sin \Omega \sin u), \\ y &= r (\sin \Omega \cos u + \cos \lambda \cos \Omega \sin u), \\ z &= r \sin \lambda \sin u, \end{aligned} \quad (7)$$

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<sup>3</sup>It is an angle in the reference  $\{xy\}$  plane from the reference  $x$  direction to the line of the nodes.

<sup>4</sup>It is the intersection of the orbital plane with the reference plane  $\{xy\}$ .

while the cartesian components of the velocity can be obtained as

$$\begin{aligned} v_x &= \frac{\partial x}{\partial f} \frac{df}{dt}, \\ v_y &= \frac{\partial y}{\partial f} \frac{df}{dt}, \\ v_z &= \frac{\partial z}{\partial f} \frac{df}{dt}, \end{aligned} \tag{8}$$

in which  $df/dt$  is given by eq. (11).

Then,  $A_R, A_T, A_N$  are to be plunged into the right-hand-sides of the Gauss equations for the variations of the Keplerian orbital elements. They are (Roy 2005; Soffel 1989)

$$\begin{aligned} \frac{da}{dt} &= \frac{2}{n\sqrt{1-e^2}} [A_R e \sin f + A_T \left(\frac{p}{r}\right)], \\ \frac{de}{dt} &= \frac{\sqrt{1-e^2}}{na} \{A_R \sin f + A_T [\cos f + \frac{1}{e} (1 - \frac{r}{a})]\}, \\ \frac{d\lambda}{dt} &= \frac{1}{na\sqrt{1-e^2}} A_N \left(\frac{r}{a}\right) \cos u, \\ \frac{d\Omega}{dt} &= \frac{1}{na\sqrt{1-e^2} \sin \lambda} A_N \left(\frac{r}{a}\right) \sin u, \\ \frac{d\omega}{dt} &= -\cos \lambda \frac{d\Omega}{dt} + \frac{\sqrt{1-e^2}}{nae} \left[-A_R \cos f + A_T \left(1 + \frac{r}{p}\right) \sin f\right], \\ \frac{d\mathcal{M}}{dt} &= n - \frac{2}{na} A_R \left(\frac{r}{a}\right) - \frac{(1-e^2)}{nae} \left[-A_R \cos f + A_T \left(1 + \frac{r}{p}\right) \sin f\right], \end{aligned} \tag{9}$$

where  $n \doteq \sqrt{GM/a^3}$  is the Keplerian mean motion related to the orbital period by  $n = 2\pi/P_b$ .

As explained in the Introduction, the right-hand-sides of eq. (9), computed for the perturbing accelerations of the dynamical effect considered, have to be inserted into the analytic expression of the time variation  $dY/dt$  of the observable  $Y$  of interest which, then, must be averaged over one orbital revolution according to eq. (2) by means of (Roy 2005)

$$df = \left(\frac{a}{r}\right)^2 \sqrt{1-e^2} d\mathcal{M}, \tag{10}$$

and

$$dt = \frac{(1-e^2)^{3/2}}{n(1+e \cos f)^2} df. \tag{11}$$

## 2.2. The effect of general relativity

In its slow-motion and weak-field approximation, GTR predicts that a slowly rotating central body of mass  $M$  and proper angular momentum  $\mathbf{L}$  induces two kinds of small perturbations on the otherwise Keplerian orbital motion of a test particle. The largest one is dubbed gravitoelectric (GE) (Mashhoon 2007), and depends only on the mass  $M$  of the body which acts as source of the gravitational field. It is responsible of the well-known anomalous secular precession of the perihelion of Mercury of  $43.98 \text{ arcsec cty}^{-1}$  in the field of the Sun. There is also a smaller perturbation, known as gravitomagnetic (GM) (Mashhoon 2007), which depends on  $\mathbf{L}$ : it causes the Lense-Thirring (Lense & Thirring 1918) precessions of the node and pericenter of a test particle. In the linearized gravitoelectromagnetic (GEM) approximation, the general relativistic perturbing acceleration  $\mathbf{A}_{\text{GTR}}$  to be added to the Newtonian monopole  $\mathbf{A}_{\text{Newton}}$  is (Soffel 1989)

$$\mathbf{A}_{\text{GTR}} = -\mathbf{E}_g - 2 \left( \frac{\mathbf{v}}{c} \right) \times \mathbf{B}_g, \quad (12)$$

with

$$\begin{aligned} \mathbf{E}_g &= -\frac{GM}{c^2 r^3} \left[ \left( \frac{4GM}{r} - v^2 \right) \mathbf{r} + 4 (\mathbf{r} \cdot \mathbf{v}) \mathbf{v} \right], \\ \mathbf{B}_g &= -\frac{G}{cr^3} [\mathbf{L} - 3 (\mathbf{L} \cdot \hat{\mathbf{r}}) \hat{\mathbf{r}}], \end{aligned} \quad (13)$$

where  $c$  denotes the speed of light in vacuum. In eq. (12)-eq. (13)  $\mathbf{E}_g$  is the GE field, while  $\mathbf{B}_g$  is the GM one. In regard to  $\mathbf{B}_g$ , in the case of a rotating BH, the existence of the horizon in the Kerr (1963) metric, which describes the spacetime outside it, implies a maximum value for its angular momentum (Bardeen et al. 1972; Melia et al. 2001)

$$L_{\bullet}^{(\text{max})} = \frac{M_{\bullet}^2 G}{c}. \quad (14)$$

If such a limit is actually reached or not by astrophysical BHs depends on their accretion history (Bardeen 1970). In fact, recent measurements of the spin of the SBH in Sgr A\* obtained in the context of discseismology by means of newly detected quasi-periodic oscillations (QPOs) of radio emission point towards (Kato et al. 2010)

$$\chi_{\bullet} \doteq \frac{L_{\bullet} c}{M_{\bullet}^2 G} = 0.44 \pm 0.08. \quad (15)$$

Genzel et al. (2003) showed that for Sgr A\* the spin parameter is  $0.52 \pm (0.1, 0.08, 0.08)$  or larger, if a QPO observed from Sgr A\* in 2003 is of dynamical origin. On the other hand, it must be noted that X-ray QPOs oscillations are rather disputed. Actually, in the case of the S stars orbiting the SBH in Sgr A\*, while the weak-field approximation is acceptable since for S2

$$\frac{\langle U \rangle}{c^2} = \frac{GM_{\bullet}}{c^2} \left\langle \frac{1}{r} \right\rangle = \frac{GM_{\bullet}}{ac^2} = 4 \times 10^{-5}, \quad (16)$$



the slow-motion approximation may be, in principle, less adequate. Indeed, the speed of, say, S2 at perinigricon is as large as 2.6% of the speed of light, while at apogricon it is 0.1% of  $c$ . Thus, in the dynamical equation of motion of eq. (12) higher order relativistic corrections should be, in principle, taken into account (Mashhoon 2005; Capozziello et al. 2009). Anyway, as we will see, they induce negligible consequences on the dynamical effects we are interested in, given the present-day level of accuracy in S stars spectroscopy.

### 2.2.1. The gravitoelectric, Schwarzschild-like perturbation

By defining

$$\mathcal{R}_g \doteq \frac{GM}{c^2}, \quad (17)$$

the  $R - T - N$  components of the general relativistic GE perturbing acceleration  $\mathbf{E}_g$  are (Soffel 1989)

$$\begin{aligned} A_R^{(\text{GE})} &= \frac{n^2 \mathcal{R}_g}{(1-e^2)^3} (1 + e \cos f)^2 (3 + 2e \cos f - e^2 + 4e^2 \sin^2 f), \\ A_T^{(\text{GE})} &= \frac{n^2 \mathcal{R}_g}{(1-e^2)^3} (1 + e \cos f)^2 4e \sin f (1 + e \cos f), \\ A_N^{(\text{GE})} &= 0. \end{aligned} \quad (18)$$

Note that  $[\mathcal{R}_g] = \text{L}$ , so that  $[n^2 \mathcal{R}_g] = \text{L T}^{-2}$ . Moreover, eq. (18) does not depend on the inclination of the orbit to the plane of the sky.

### 2.2.2. The gravitomagnetic, Lense-Thirring-like perturbation

The  $R - T - N$  components of the Lorentz-like general relativistic GM perturbing acceleration induced by the rotation of the central body with proper angular momentum  $L$  are (Soffel 1989)

$$\begin{aligned} A_R^{(\text{GM})} &= \eta_g \cos \Psi (1 + e \cos f), \\ A_T^{(\text{GM})} &= -\eta_g e \cos \Psi \sin f, \\ A_N^{(\text{GM})} &= \eta_g \sin \Psi (1 + e \cos f) \left[ 2 \sin u + e \left( \frac{\sin f \cos u}{1 + e \cos f} \right) \right], \end{aligned} \quad (19)$$

with

$$\eta_g \doteq \frac{\xi_g n}{a^2 (1 - e^2)^{7/2}} (1 + e \cos f)^3, \quad (20)$$

and

$$\xi_g \doteq \frac{2GL}{c^2}. \quad (21)$$

Note that  $[\xi_g] = \text{L}^3 \text{T}^{-1}$ , so that  $[\eta_g] = \text{L} \text{T}^{-2}$ . In eq. (19)  $\Psi$  is the angle between the orbital plane and the equatorial plane of the central body, i.e.  $\mathbf{L}$  has been assumed directed along the positive  $z$  axis so that the reference  $\{xy\}$  plane coincides with the equatorial plane of the central body ( $\lambda \rightarrow \Psi$ ). For equatorial orbits, i.e. for  $\Psi = 0$ ,  $A_N^{(\text{GM})} = 0$  and  $A_R^{(\text{GM})} \neq 0, A_T^{(\text{GM})} \neq 0$ . Instead, for polar orbits, i.e. for  $\Psi = 90$  deg, only the normal component does not vanish.

### 2.3. The quadrupole mass moment

The external gravitational field of a rotating body undergoes departures from spherical symmetry because of the distortion of its shape due to the resulting centrifugal force. An oblate body of equatorial radius  $R_e$  and adimensional quadrupole mass moment  $J_2$  affects the orbital motion of a test particle with a non-central perturbing acceleration (Cunningham 1970; Vrbik 2005)

$$\mathbf{A}^{(J_2)} = -\frac{3J_2 R_e^2 GM}{2r^4} \left\{ \left[ 1 - 5(\hat{\mathbf{r}} \cdot \hat{\mathbf{L}})^2 \right] \hat{\mathbf{r}} + 2(\hat{\mathbf{r}} \cdot \hat{\mathbf{L}}) \hat{\mathbf{L}} \right\}, \quad (22)$$

where  $\hat{\mathbf{L}}$  is the unit vector of the body's angular momentum, directed here along the positive  $z$  axis so that the equatorial plane is the reference  $\{xy\}$  plane (again,  $\lambda \rightarrow \Psi$ ). According to eq. (3)-eq. (5) and eq. (7), the  $R - T - N$  components of eq. (22) are

$$\begin{aligned} A_R^{(J_2)} &\doteq \mathbf{A}^{(J_2)} \cdot \hat{\mathbf{R}} = -\frac{3n^2 R_e^2 J_2}{8a(1-e^2)^4} (1 + e \cos f)^4 (1 + 3 \cos 2\Psi + 6 \sin^2 \Psi \cos 2u), \\ A_T^{(J_2)} &\doteq \mathbf{A}^{(J_2)} \cdot \hat{\mathbf{T}} = -\frac{3n^2 R_e^2 J_2}{2a(1-e^2)^4} (1 + e \cos f)^4 \sin^2 \Psi \sin 2u, \\ A_N^{(J_2)} &\doteq \mathbf{A}^{(J_2)} \cdot \hat{\mathbf{N}} = -\frac{3n^2 R_e^2 J_2}{2a(1-e^2)^4} (1 + e \cos f)^4 \sin 2\Psi \sin u. \end{aligned} \quad (23)$$

Note that  $[n^2 R_e^2 a^{-1}] = \text{L} \text{T}^{-2}$ . For  $\Psi = 0$ , i.e. for equatorial orbits, only the radial component is not zero. For polar orbits, i.e. for  $\Psi = 90$  deg, the normal component vanishes, contrary to the radial and transverse ones.

Also a rotating BH should have a quadrupole mass moment, so that it affects the orbital motion of a distant test particle with a perturbing acceleration analogous to that of eq. (22). It is customary to introduce a dimensional quadrupole parameter  $Q_2$ ,  $[Q_2] = \text{L}^5 \text{T}^{-2}$ , in such a way that (Will 2008)

$$J_2 R_e^2 GM \rightarrow Q_2 \quad (24)$$

throughout eq. (22) and eq. (23). According to the “no-hair” or uniqueness theorems of GTR (Chrusciel 1994; Heusler 1998), an electrically neutral BH is completely characterized by its mass  $M_\bullet$  and angular momentum  $L_\bullet$  only. As a consequence, all the multipole moments of its external spacetime are functions of  $M_\bullet$  and  $L_\bullet$ . In particular, the quadrupole mass moment is

$$Q_2^\bullet = -\frac{L_\bullet^2 G}{c^2 M_\bullet} = -\chi_\bullet^2 \frac{G^3 M_\bullet^3}{c^4}. \quad (25)$$

Thus, in the case of the SBH in GC eq. (15) yields

$$|Q_2^\bullet| = 3.585 \times 10^{45} \text{ m}^5 \text{ s}^{-2}. \quad (26)$$

In terms of the adimensional coefficient  $J_2$ , by assuming the Schwarzschild radius  $r_g$  for the equatorial radius  $R_e$  of the BH, eq. (26) would correspond to

$$|J_2^\bullet| = 4 \times 10^{-2}. \quad (27)$$

Just for the sake of a comparison, for the Sun, Jupiter and the Earth we have  $J_2^\odot = 2 \times 10^{-7}$  from helioseismology (Mecheri et al. 2004),  $J_2^{(\text{Jup})} = 1.46 \times 10^{-2}$  from the flybys of some spacecraft (Jacobson 2003),  $J_2^\oplus = 1.08 \times 10^{-3}$  from the dedicated GRACE spacecraft (Bruinsma 2010), respectively.

## 2.4. Inner diffuse mass distribution

It is well recognized that, in addition to the dynamical effects considered so far directly related to the SBH, also the impact of a diffuse cluster of non-luminous ordinary matter<sup>5</sup> around the BH due to massive remnants of various kinds (Morris 1993) should be taken into account. Indeed, reasoning in terms of the perinigricon, such an extended material component would induce a retrograde precession  $\dot{\omega}_\bullet$  which may overwhelm the general relativistic GE one for certain values of its mass (Rubilar & Eckart 2001). Let us, now, work out in detail its dynamical effects.

Following Rubilar & Eckart (2001), Mouawad et al. (2005), Gillessen et al. (2009b), we adopt a Plummer density profile

$$\varrho_{\text{dm}}(r) = \frac{3\mu M_\bullet}{4\pi d_c^3} \left(1 + \frac{r^2}{d_c^2}\right)^{-5/2}, \quad (28)$$

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<sup>5</sup>In principle, one should also take into account the issue of non-baryonic dark matter cusps in relation to the merging history of galaxies (Merritt & Milosavljević 2002). It is beyond the scopes of this work.

where the core radius is (Gillessen et al. 2009b)

$$d_c = 15 \text{ mpc}, \quad (29)$$

in agreement with the observed light profile (Mouawad et al. 2005), while  $\mu$ , the mass parameter, is the ratio of the total extended mass  $M_{\bar{r}}$  at a given distance  $\bar{r}$  to the central point mass. Gillessen et al. (2009b), with a fit involving S2 able to probe the mass enclosed between its apogricon and perigricon, yield

$$\mu \leq 0.04. \quad (30)$$

It should be mentioned that Mouawad et al. (2005), with the position of Sgr A\* as a free discrete input parameter, provided for the first time an upper limit of

$$\mu \leq 0.05. \quad (31)$$

From the Poisson equation

$$\nabla^2 \mathcal{U} = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \mathcal{U}}{\partial r} \right) = 4\pi G \varrho \quad (32)$$

for the gravitational potential  $\mathcal{U}$ , written in spherical coordinates since  $\mathcal{U} = \mathcal{U}(r)$  in view of the fact that  $\varrho = \varrho(r)$ , the perturbing acceleration

$$\mathbf{A}_{(\text{dm})} = -\nabla \mathcal{U}_{(\text{dm})} = -\frac{\partial \mathcal{U}_{(\text{dm})}}{\partial r} \hat{\mathbf{r}} \quad (33)$$

easily follows. It turns out

$$\begin{aligned} A_R^{(\text{dm})} &= -\frac{GM_\bullet \mu}{d_c^3} \left( 1 + \frac{r^2}{d_c^2} \right)^{-3/2} r, \\ A_T^{(\text{dm})} &= 0, \\ A_N^{(\text{dm})} &= 0. \end{aligned} \quad (34)$$

### 3. The radial velocity

The basic observable in spectroscopic studies of binary systems is the radial velocity  $v_\rho$ , i.e. the component of the velocity vector  $\mathbf{v}$  of one of (or both) the system's partners along the line-of-sight whose unit vector  $\hat{\boldsymbol{\rho}}$  has been assumed directed along the  $z$  axis. Concerning the S stars orbiting the Galactic SBH, let us mention the possibility that some of them may actually be binaries themselves. In this case, additional “noise” to the radial velocity data would be introduced. Its expression for unperturbed, Keplerian elliptic

orbits, up to the velocity of the system’s center of mass  $v_0$ , can be obtained by using the  $z$  components of eq. (3) and eq. (4) with  $\lambda \rightarrow i$ , and recalling that, for a Keplerian orbit (Roy 2005),

$$\mathbf{v} = v_R \hat{\mathbf{R}} + v_T \hat{\mathbf{T}} = \frac{na}{\sqrt{1-e^2}} \left[ e \sin f \hat{\mathbf{R}} + (1 + e \cos f) \hat{\mathbf{T}} \right]. \quad (35)$$

The result is

$$v_\rho = K [e \cos \omega + \cos(f + \omega)], \quad (36)$$

where  $2K$  is the total observed range of radial velocity defined by

$$K \doteq \frac{na \sin i}{\sqrt{1-e^2}}. \quad (37)$$

Note that eq. (36) and eq. (37) agree with the expressions given by, e.g., Batten (2001) and Padmanabhan (2010).

Perturbing dynamical effects affect the radial velocity as well by inducing, in principle, a non-vanishing net radial acceleration over one orbital period. It can straightforwardly be worked out from eq. (2) with  $Y \rightarrow v_\rho$  by noting that, in this case, the perturbations of all the six Keplerian orbital elements are involved. In this respect, a special care is required for the parameter  $i$  entering eq. (37). It is the angle between the unit vector  $\hat{\boldsymbol{\ell}}$  of the orbital angular momentum and the unit vector  $\hat{\boldsymbol{\rho}}$  of the line-of-sight pointing towards the observer. From the spherical law of cosines (Gellert et al. 1989; Zwillinger 1995)

$$\cos B = \sin C \sin A \cos b - \cos C \cos A \quad (38)$$

with the identifications<sup>6</sup>  $A \rightarrow \Psi, B \rightarrow \pi - i, C \rightarrow I, b \rightarrow \pi - \Omega$ , it turns out

$$\cos i = \sin \Psi \sin I \cos \Omega + \cos \Psi \cos I, \quad (39)$$

where  $\Psi$  is the angle between  $\hat{\boldsymbol{\ell}}$  and the unit vector  $\hat{\mathbf{L}}$  of the central body’s proper angular momentum,  $I$  is the angle between  $\hat{\mathbf{L}}$  and  $\hat{\boldsymbol{\rho}}$ , and  $\Omega$  is the longitude of the ascending node defined from

$$\sin \Psi \sin I \cos \Omega = (\hat{\mathbf{L}} \times \hat{\boldsymbol{\ell}}) \cdot (\hat{\mathbf{L}} \times \hat{\boldsymbol{\rho}}). \quad (40)$$

From eq. (39) it turns out

$$\sin i \left( \frac{di}{dt} \right) = (\sin \Psi \cos I - \cos \Psi \sin I \cos \Omega) \frac{d\Psi}{dt} + \sin \Psi \sin I \sin \Omega \left( \frac{d\Omega}{dt} \right). \quad (41)$$

As eq. (9) shows for  $\lambda \rightarrow \Psi$ , only those perturbing accelerations, like the general relativistic GE one, having no out-of-plane components  $A_N$  leave  $i$  unaffected since both  $\Psi$  and  $\Omega$  may

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<sup>6</sup>In such a way,  $\Omega$  results to be prograde with respect to the orbital motion, i.e.  $\Omega$  follows it, coherently with the definition of the longitude of ascending node. Moreover, it lies in the equatorial plane.

change due to  $A_N$ . Thus, in the following expressions for the effects due to the general relativistic GM field and  $Q_2$  it must be recalled that  $\Omega$  is intended to be referred to the equatorial plane, not to the plane of the sky. It must also be noted that both  $I$  and  $\Psi$  are unknown.

### 3.1. The effect of general relativity

General relativity dynamically affects the radial velocity of non-circular orbits by causing averaged long-term variations of it. Indeed, an exact calculation in  $e$  with eq. (18) yields

$$\langle \dot{v}_\rho^{(\text{GE})} \rangle = (n^2 \mathcal{R}_g) \frac{15e(1+e^2) \sin i \sin \omega}{8(1-e^2)^{5/2}}, \quad (42)$$

for the Schwarzschild-type, gravitoelectric component, while the secular acceleration due to the Lense-Thirring-type, gravitomagnetic terms of eq. (19) is<sup>7</sup>

$$\langle \dot{v}_\rho^{(\text{GM})} \rangle = \left( \frac{nGL}{c^2 a^2} \right) \frac{e}{4(1-e^2)^2} [\mathcal{V}_c(i, I, \Omega, \Psi) \cos \omega + \mathcal{V}_s(i, I, \Omega, \Psi) \sin \omega], \quad (43)$$

with

$$\begin{aligned} \mathcal{V}_c &\doteq 11 \cot i \sin I \sin \Psi \sin \Omega, \\ \mathcal{V}_s &\doteq \frac{\csc i}{4} \{ \cos \Omega \sin 2I (\sin \Psi - \sin 3\Psi) - \\ &\quad - \sin^2 \Psi \cos \Psi [\cos 2I (3 + \cos 2\Omega) + 2 \sin^2 \Omega] + 104 \sin^2 i \cos \Psi \}. \end{aligned} \quad (44)$$

Let us recall again that  $I$  is the angle-unknown-between  $\mathbf{L}$  and the line-of-sight, and  $\Omega$  lies in the equatorial plane of the source: for it the values determined in literature (Ghez et al. 2008; Gillessen et al. 2009a,b) cannot be used because they refer to the plane of the sky<sup>8</sup>. Also  $\Psi$  is unknown. Anyway, in order to give an order-of-magnitude evaluation of the Lense-Thirring long-term radial acceleration we will simply look at the multiplicative dimensional factor in front of the square brackets in eq. (43). For  $e \rightarrow 0$  both eq. (42) and eq. (43) vanish. The general relativistic effects are non-vanishing either for edge-on ( $i = 90$  deg) or equatorial orbits ( $\Psi = 0$ ), with eq. (42) which is independent of  $\Psi$ , contrary to eq. (43).

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<sup>7</sup> It is assumed that the orbital motion is prograde with respect to the BH's spin; a – sign would occur in the opposite case.

<sup>8</sup>In principle, spherical trigonometric formulas relate both the nodes  $\Omega_e$  and  $\Omega_s$ , where e and s denotes the equator and the sky. However, the knowledge of the angle, in the plane of the sky, between the equatorial plane and the reference  $\{xz\}$  plane would be required.

Some preliminary and approximate calculation of the effects of both the general relativistic GEM dynamical effects on the velocity of S stars orbiting the SBH in GC can be found in Kannan & Saha (2009); they do neither deal with the directly measurable radial velocity nor with its possible variations. See also Angélil & Saha (2010) for GEM effects on the traveling photons paths.

Special relativistic effects on the radial velocity related to the Doppler effect have been considered by Zucker et al. (2006); anyway, they do not involve net variations of  $v_\rho$  over one orbital revolution.

### 3.2. The quadrupole mass moment

The BH's oblateness causes an averaged long-term variation of the radial velocity only if the orbit is elliptic. Indeed, it turns out

$$\left\langle \dot{V}_\rho^{(J_2)} \right\rangle = - \left( \frac{Q_2^\bullet}{a^4} \right) \frac{3e}{32(1-e^2)^{7/2} \sin i} [\mathcal{J}_c(e, i, I, \Omega, \Psi) \cos \omega + \mathcal{J}_s(e, i, I, \omega, \Omega, \Psi) \sin \omega], \quad (45)$$

with

$$\begin{aligned} \mathcal{J}_c &\doteq 10(1-e^2) \cos i \sin I \sin 2\Psi \sin \Omega, \\ \mathcal{J}_s &\doteq 2(1-e^2) \cos i \sin 2\Psi (\cos I \sin \Psi - \sin I \cos \Psi \cos \Omega) + \\ &+ \sin^2 i [7 + 47 \cos 2\Psi + \sin^2 \Psi \cos 2\omega - \\ &- \frac{e^2}{16} (259 + 429 \cos 2\Psi - 44 \sin^2 \Psi \cos 2\omega)]. \end{aligned} \quad (46)$$

It is an exact result in  $e$ , and vanishes in the limit  $e \rightarrow 0$ . Note that, according to eq. (46), eq. (45) vanishes neither for equatorial orbits ( $\Psi = 0$ ) nor for edge-on configurations ( $i = 90$  deg). Also in this case,  $\Omega$  refers to the equatorial plane.

### 3.3. The diffuse inner dark matter

The averaged long-term effect of the diffuse inner dark matter<sup>9</sup> on  $\dot{v}_\rho$  can be worked from eq. (34). An approximate calculation with

$$\left( 1 + \frac{r^2}{d_c^2} \right)^{-3/2} \approx 1 - \frac{3}{2} \frac{r^2}{d_c^2} \quad (47)$$

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<sup>9</sup>The effects of non-baryonic dark matter on the perinigricon precession have been considered by, e.g., Zakharov et al. (2007).

and

$$(1 + e \cos f)^{-4} \approx 1 - 4e \cos f, \quad (1 + e \cos f)^{-5} \approx 1 - 5e \cos f, \quad (48)$$

yields

$$\langle \dot{v}_\rho^{(\text{dm})} \rangle = \left( \frac{GM_\bullet a}{d_c^3} \right) \frac{e(1 - e^2)^{3/2} \mu \sin i}{8} \left[ 32 - 5e^2(2 + 3e^2) + 6(1 - e^2)^2(5e^2 - 11) \frac{a^2}{d_c^2} \right] \sin \omega. \quad (49)$$

Note that eq. (49) vanishes for  $e \rightarrow 0$  and for face-on ( $i = 0$ ) orbital configurations. Concerning the validity of the approximation of eq. (47), it actually holds for S2. Indeed, its orbital parameters and eq. (29) for  $d_c$  tell us that

$$0.002 \leq \frac{r^2}{d_c^2} \leq 0.4. \quad (50)$$

#### 4. Numerical evaluations and confrontation with the measurement accuracy

By using the known orbital parameters of S2 along with the associated uncertainties (Gillessen et al. 2009b), the known mass  $M_\bullet$  of the SBH in the GC (Gillessen et al. 2009b) and the values for its angular momentum and quadrupole mass moment from eq. (15) and eq. (26), it turns out that<sup>10</sup>

$$\begin{aligned} \langle \dot{v}_\rho^{(\text{GE})} \rangle &= (8.2 \pm 2.6) \times 10^{-5} \text{ m s}^{-2}, \\ \langle \dot{v}_\rho^{(\text{dm})} \rangle &= 3.8 \times 10^{-6} \text{ m s}^{-2}, \\ \langle \dot{v}_\rho^{(\text{GM})} \rangle &\propto 1.3 \times 10^{-8} \text{ m s}^{-2}, \\ \langle \dot{v}_\rho^{(Q_2)} \rangle &\propto 1 \times 10^{-10} \text{ m s}^{-2}. \end{aligned} \quad (51)$$

Concerning a possible measurement of a net change in the radial velocity of S2 after it completed one full orbital revolution, no empirical evidences for it exist to date in literature, at least to the knowledge of this author. Anyway, measurements of the magnitude of the three-dimensional acceleration of S2 after 2 years (1997-1999) are available; its accuracy amounts to  $4 \times 10^{-4} \text{ m s}^{-2}$  (Ghez et al. 2000). By assuming an uncertainty of about  $15 \text{ km s}^{-1}$  in measuring the radial velocity of S2 (Gillessen et al. 2009a), an overall accuracy

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<sup>10</sup>The figures for  $\langle \dot{v}_\rho^{(\text{GM})} \rangle$  and  $\langle \dot{v}_\rho^{(Q_2)} \rangle$  refer to the dimensional multiplicative factors in front of the square brackets in eq. (43) and in eq. (45).



of the order of  $2.4 \times 10^{-5} \text{ m s}^{-2}$  in  $\langle \dot{v}_\rho \rangle$  may be assumed in future over an observational time span  $\Delta t = 20 \text{ yr}$ . Actually, it must be pointed out that the currently available radial velocity measurements do not yet cover one full orbit revolution for S2. Indeed, the first radial velocity data points are from 2000, then 2002; they are more densely sampled from 2003 onwards (S. Gillessen, private communication, August 2010).

It has to be pointed out that the total accuracy reachable in the changes in the radial velocities is actually impacted by the uncertainty in LSR itself. Indeed, as explained by (Ghez et al. 2008), to obtain the radial velocities with respect to the LSR, each observed radial velocity has to be corrected for the Earth’s rotation, its motion around the Sun, and the Sun’s peculiar motion with respect to the LSR (nominal value  $U_\odot = 10 \text{ km s}^{-1}$ , directed radially inwards, (Dehnen & Binney 1998)). Since the LSR is defined as the velocity of an object in circular orbit at the radius of the Sun, the Sun’s peculiar motion with respect to the average velocity of stars in its vicinity should give the Sun’s motion toward the GC. In all such a machinery, also the rotation speed  $\Theta_0$  of LSR enters; recent evaluations by Reid et al. (2009) yield an uncertainty of the order of  $16 \text{ km s}^{-1}$  corresponding to a future uncertainty of  $2 \times 10^{-5} \text{ m s}^{-2}$  over  $\Delta t = 20 \text{ yr}$ . Moreover, also the motion of the SBH itself should be taken into account (Ghez et al. 2008). In particular, the uncertainty in its radial velocity can be evaluated to be  $2 \text{ km s}^{-1}$  (Gould 2004) implying a limit in the accuracy in  $\langle \dot{v}_\rho \rangle$  of about  $1 \times 10^{-6} \text{ m s}^{-2}$  over  $\Delta t = 20 \text{ yr}$ . Such limiting factors should be taken into account when future improvements in measuring radial velocities are discussed. Searches for pulsars orbiting the SBH are currently underway (Macquart et al. 2010); their discovery may yield orbiting probes with a better accuracy in their radial velocity changes.

These considerations show that while the gravitomagnetic and the quadrupole effects are far from being directly detectable in such a way, the gravitoelectric trend lies just at the edge of the measurability capabilities. The effect of the diffuse dark matter inside the orbit of S2 is one order of magnitude smaller than the general relativistic GE one. Incidentally, such figures indicate that higher-order corrections to the computed effects of eq. (42)-eq. (43) in the linear GEM approximation due to the relativistic motion of S2 can be neglected at the moment.

Finally, in view of the possible discovery in the near future of stars closer than S2 we plot in Figure 1 the magnitudes of the effects considered as functions of the semi-major axis for different values of the eccentricity.

## 5. Conclusions

One of the directly measurable quantities of the system constituted by the S stars orbiting the Supermassive Black Hole located at the center of the Milky Way in the radio source Sgr A\* is the radial velocity. Given that S2 has already completed one full orbital revolution, with a period of 15.9 yr, since its discovery, and in view of the possible detection

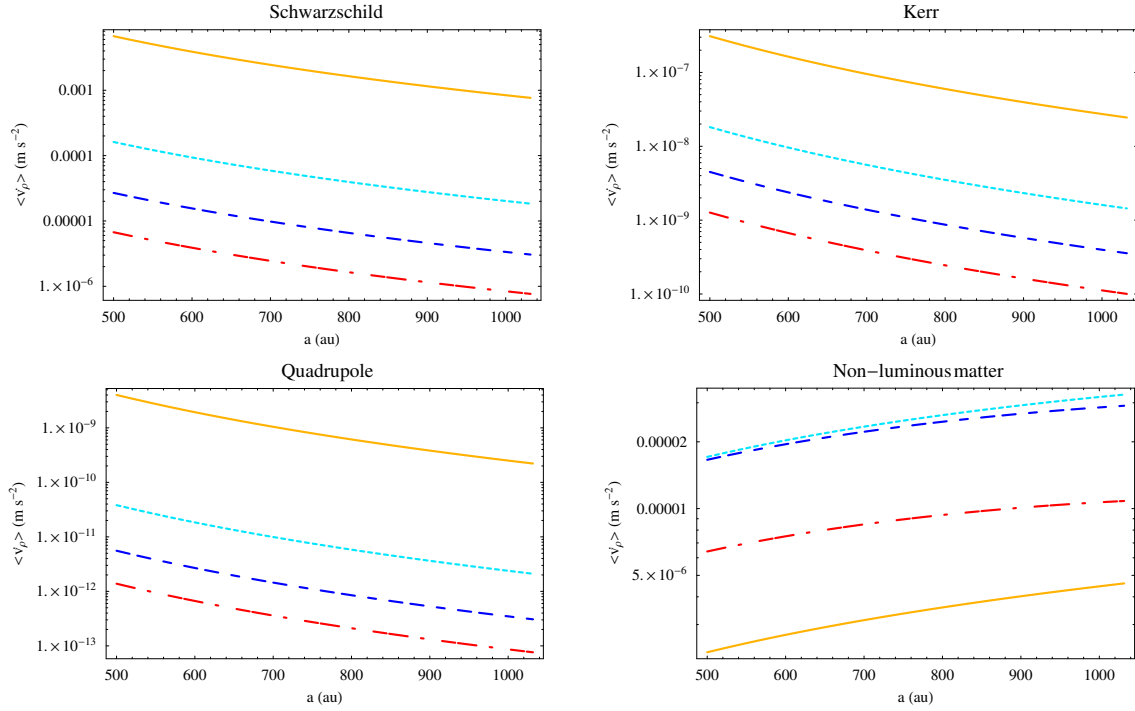


Fig. 1.— Maximum values of the long-term time variations  $\langle \dot{v}_\rho \rangle$ , in  $\text{m s}^{-2}$ , as a function of  $a$  ( $500 \text{ au} \leq a \leq 1031.69 \text{ au}$ ) for different values of the eccentricity:  $e = 0.1$  (red dash-dotted line),  $e = 0.3$  (blue dashed line),  $e = 0.6$  (light blue dotted line),  $e = 0.9$  (yellow continuous line). For the the spin  $L_\bullet$  and the quadrupole  $Q_2^\bullet$  of the SBH we used the values of eq. (15) and eq. (26), respectively.

in the near future of other stars and pulsars with shorter orbital periods, we looked at the cumulative, long term time variations of the radial velocity caused by several Newtonian and Einsteinian dynamical effects. They are both the general relativistic Schwarzschild and Kerr-like components of the spacetime metric, the quadrupole mass moment and the diffuse dark mass distribution made by stellar remnants enclosed within the star’s orbit. We analytically worked out the long-term variations in the radial velocity induced by them by finding non-zero effects for all of them. We used S2 for numerically computing their magnitudes. They are  $8 \times 10^{-5} \text{ m s}^{-2}$  (Schwarzschild),  $4 \times 10^{-6} \text{ m s}^{-2}$  (dark matter),  $1 \times 10^{-8} \text{ m s}^{-2}$  (Kerr),  $1 \times 10^{-10} \text{ m s}^{-2}$  (quadrupole), respectively. In computing the general relativistic variations of the radial velocity, we remained within the post-Newtonian regime by neglecting relativistic corrections of higher order in the equations of motion. The figures for the the Kerr and quadrupole effects have been computed by using the latest determinations of the angular momentum parameter of the Galactic black hole, and in the “no-hair” hypothesis, respectively. For the dark matter distribution we used a Plummer-like mass density profile. By assuming a present-day uncertainty of about  $15 \text{ km s}^{-1}$  in the radial velocity measurements, its time changes may be detected in the future at a  $\approx 10^{-5} \text{ m s}^{-2}$  level over an observational time span of 20 yr; at present, radial velocity data cover just 7 yr. Even if such evaluations will turn out to be not too optimistic, a detection of the Kerr and the quadrupole-induced cumulative changes of the radial velocity seems to be unfeasible.

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